

Distance-regular graphs with valency k having smallest eigenvalue at most $-k/2$

Jack Koolen*

School of Mathematical Sciences
University of Science and Technology of China,
Wen-Tsun Wu Key Laboratory of the Chinese Academy of Sciences,
230026, Anhui, PR China
e-mail: koolen@ustc.edu.cn

Zhi Qiao

School of Mathematical Sciences
University of Science and Technology of China,
230026, Anhui, PR China
email: gesec@mail.ustc.edu.cn

July 23, 2015

1 Introduction

In this paper, we study the non-bipartite distance-regular graphs with valency k and having a smallest eigenvalue at most $-k/2$ (For notations and explanation of the graphs, see next section and [2] or [12]). There are seven infinite families known, namely

1. The odd polygons with valency 2;
2. The complete tripartite graphs $K_{t,t,t}$ with valency $2t$ at least 2;
3. The folded $(2D+1)$ -cubes with valency $2D+1$ and diameter $D \geq 2$;
4. The Odd graphs with valency k at least 3;
5. The Hamming graphs $H(D, 3)$ with valency $2D$ where $D \geq 2$;
6. The dual polar graphs of type $B_D(2)$ with $D \geq 2$;

*JHK was partially supported by the National Natural Science Foundation of China (No. 11471009).

7. The dual polar graphs of type ${}^2A_{2D-1}(2)$ with $D \geq 2$.

First we will show a valency bound for distance-regular graphs with a relatively large, in absolute value, smallest eigenvalue.

Theorem 1.1 *For any real number $1 > \alpha > 0$ and any integer $D \geq 2$, the number of coconnected non-bipartite distance-regular graphs with valency k at least two and diameter D , having smallest eigenvalue θ_{\min} not larger than $-\alpha k$, is finite.*

Remarks. (i) Note that the regular complete t -partite graphs $K_{t \times s}$ (s, t positive integers at least 2) with valency $k = (t-1)s$ have smallest eigenvalue $-s = -k/(t-1)$.

(ii) Note that there are infinitely many bipartite distance-regular graphs with diameter 3, for example the point-block incidence graphs of a projective plane of order q , where q is a prime power. For diameter 4 this is also true, for example the Hadamard graphs.

(iii) The second largest eigenvalue for a distance-regular graphs behaves quite differently from its smallest eigenvalue. For example $J(n, t)$ $n \geq 2t \geq 4$, has valency $t(n-t)$ and second largest eigenvalue $(n-t-1)(t-1) - 1$. So for fixed t there are infinitely many Johnson graphs $J(n, t)$ with second largest eigenvalue larger than $k/2$.

Then we classify the non-bipartite distance-regular graphs with diameter at most 4 with valency k having smallest eigenvalue at most $-k/2$ where for diameter 4 we have also the condition $a_1 \neq 0$.

Theorem 1.2 *Let Γ be a non-bipartite distance-regular graph with diameter D at most 4 where, if $D = 4$, then $a_1 \neq 0$, and valency k at least 2, having smallest eigenvalue at most $-k/2$. Then Γ is one of the following graphs:*

1. Diameter equals 1:

(a) The triangle with intersection array $\{2; 1\}$

2. Diameter equals 2:

(a) The pentagon with intersection array $\{2, 1; 1, 1\}$;

(b) The Petersen graph with intersection array $\{3, 2; 1, 1\}$;

(c) The folded 5-cube with intersection array $\{5, 4; 1, 2\}$;

(d) The 3×3 -grid with intersection array $\{4, 2; 1, 2\}$;

(e) The generalized quadrangle $GQ(2, 2)$ with intersection array $\{6, 4; 1, 3\}$;

(f) The generalized quadrangle $GQ(2, 4)$ with intersection array $\{10, 8; 1, 5\}$;

(g) A complete tripartite graph $K_{t,t,t}$ with $t \geq 2$, with intersection array $\{2t, t-1; 1, 2t\}$;

3. Diameter equals 3:

(a) The 7-gon, with intersection array $\{2, 1, 1; 1, 1, 1\}$;

(b) The Odd graph with valency 4, O_4 , with intersection array $\{4, 3, 3; 1, 1, 2\}$;

- (c) The Sylvester graph with intersection array $\{5, 4, 2; 1, 1, 4\}$;
- (d) The second subconstituent of the Hoffman-Singleton graph with intersection array $\{6, 5, 1; 1, 1, 6\}$;
- (e) The Perkel graph with intersection array $\{6, 5, 2; 1, 1, 3\}$;
- (f) The folded 7-cube with intersection array $\{7, 6, 5; 1, 2, 3\}$;
- (g) A possible distance-regular graph with intersection array $\{7, 6, 6; 1, 1, 2\}$;
- (h) A possible distance-regular graph with intersection array $\{8, 7, 5; 1, 1, 4\}$;
- (i) The truncated Witt graph associated with M_{23} (see [2, Thm 11.4.2]) with intersection array $\{15, 14, 12; 1, 1, 9\}$;
- (j) The coset graph of the truncated binary Golay code with intersection array $\{21, 20, 16; 1, 2, 12\}$;
- (k) The line graph of the Petersen graph with intersection array $\{4, 2, 1; 1, 1, 4\}$;
- (l) The generalized hexagon $GH(2, 1)$ with intersection array $\{4, 2, 2; 1, 1, 2\}$;
- (m) The Hamming graph $H(3, 3)$ with intersection array $\{6, 4, 2; 1, 2, 3\}$;
- (n) One of the two generalized hexagons $GH(2, 2)$ with intersection array $\{6, 4, 4; 1, 1, 3\}$;
- (o) One of the two distance-regular graphs with intersection array $\{8, 6, 1; 1, 3, 8\}$ (see [2, p. 386]);
- (p) The regular near hexagon $B_3(2)$ with intersection array $\{14, 12, 8; 1, 3, 7\}$;
- (q) The generalized hexagon $GH(2, 8)$ with intersection array $\{18, 16, 16; 1, 1, 9\}$;
- (r) The regular near hexagon on 729 vertices related to the extended ternary Golay code with intersection array $\{24, 22, 20; 1, 2, 12\}$;
- (s) The Witt graph associated to M_{24} (see [2, Thm 11.4.1]) with intersection array $\{30, 28, 24; 1, 3, 15\}$;
- (t) The regular near hexagon ${}^2A_5(2)$ with intersection array $\{42, 40, 32; 1, 5, 21\}$.

4. Diameter equals 4 and $a_1 \neq 0$;

- (a) The generalized octagon $GO(2, 1)$ with intersection array $\{4, 2, 2, 2; 1, 1, 1, 2\}$;
- (b) The distance-regular graph with intersection array $\{6, 4, 2, 1; 1, 1, 4, 6\}$ (see [2, Thm 13.2.1]);
- (c) The Hamming graph $H(4, 3)$ with intersection array $\{8, 6, 4, 2; 1, 2, 3, 4\}$;
- (d) A generalized octagon $GO(2, 4)$ with intersection array $\{10, 8, 8, 8; 1, 1, 1, 5\}$;
- (e) The Cohen-Tits regular near octagon associated with the Hall-Janko group (see [2, Thm 13.6.1]) with intersection array $\{10, 8, 8, 2; 1, 1, 4, 5\}$.
- (f) The regular near hexagon $B_4(2)$ with intersection array $\{30, 28, 24, 16; 1, 3, 7, 15\}$;
- (g) The regular near hexagon ${}^2A_7(2)$ with intersection array $\{170, 168, 160, 128; 1, 5, 21, 85\}$.

Remark. It is not known whether the generalized octagon $GO(2, 4)$ with intersection array $\{10, 8, 8, 8; 1, 1, 1, 5\}$ is unique.

This result is an extension of De Bruyn's results [4, Sects. 3.5 & 3.6] on regular near hexagons and octagons, with lines with size 3, see also Theorem 6.2.

As a consequence of Theorem 1.2, we also obtain a complete classification of the 3-chromatic distance-regular graphs with diameter 3 and the 3-chromatic distance-regular graphs with diameter 4 and intersection number $a_1 \neq 0$.

Theorem 1.3 (i) Let Γ be a 3-chromatic distance-regular graph with diameter 3. Then Γ is one of the following:

1. The 7-gon, with intersection array $\{2, 1, 1; 1, 1, 1\}$;
2. The Odd graph with valency 4, O_4 , with intersection array $\{4, 3, 3; 1, 1, 2\}$;
3. The Perkel graph with intersection array $\{6, 5, 2; 1, 1, 3\}$;
4. The generalized hexagon $GH(2, 1)$ with intersection array $\{4, 2, 2; 1, 1, 2\}$;
5. The Hamming graph $H(3, 3)$ with intersection array $\{6, 4, 2; 1, 2, 3\}$;
6. The regular near hexagon on 729 vertices related to the extended ternary Golay code with intersection array $\{24, 22, 20; 1, 2, 12\}$.

(ii) Let Γ be a 3-chromatic distance-regular graph with diameter 4 and $a_1 \neq 0$. Then Γ is the Hamming graph $H(4, 3)$ with intersection array $\{8, 6, 4, 2; 1, 2, 3, 4\}$, or the generalized hexagon $GO(2, 1)$ with intersection array $\{4, 2, 2, 2; 1, 1, 1, 2\}$.

This result is an extension of Blokhuis et al. [1]. In that paper, they determined all the 3-chromatic distance-regular graphs among the known examples.

This paper is organised as follows, in Section 2 we give definitions and preliminaries, and in Section 3 we give the proof of the valency bound, Theorem 1.1. In Section 4, we treat the strongly regular graphs. In Section 5 we give a bound on the intersection number c_2 . In Sections 6 we treat the case $a_1 = 1$ and in Section 7 we treat the case $a_1 = 0$. In Section 8, we give the proofs of Theorems 1.2 and 1.3. In the last section we give some open problems.

2 Preliminaries and definitions

All graphs considered in this paper are finite, undirected and simple (for more background information, see [2] or [12]). For a connected graph $\Gamma = (V(\Gamma), E(\Gamma))$, the distance $d(x, y)$ between any two vertices x, y is the length of a shortest path between x and y in Γ , and the diameter D is the maximum distance between any two vertices of Γ . For any vertex x , let $\Gamma_i(x)$ be the set of vertices in Γ at distance precisely i from x , where $0 \leq i \leq D$. For a set of vertices x_1, \dots, x_n , let $\Gamma_1(x_1, \dots, x_n)$ denote $\cap_{i=1}^n \Gamma_1(x_i)$. For a non-empty subset $S \subseteq V(\Gamma)$, $\langle S \rangle$ denotes the induced subgraph on S . A graph is coconnected if its complement is connected. A connected graph Γ with diameter D is called a *distance-regular graph* if there are integers b_i, c_i ($0 \leq i \leq D$) such that for any two vertices $x, y \in V(\Gamma)$ with $i = d(x, y)$, there are exactly c_i neighbors of y in $\Gamma_{i-1}(x)$ and b_i neighbors of y in $\Gamma_{i+1}(x)$. The numbers b_i, c_i and $a_i := b_0 - b_i - c_i$ ($0 \leq i \leq D$) are called the *intersection numbers* of Γ . Set $c_0 = b_D = 0$. We observe $a_0 = 0$ and $c_1 = 1$. The array $\iota(\Gamma) = \{b_0, b_1, \dots, b_{D-1}; c_1, c_2, \dots, c_D\}$ is called the *intersection array* of Γ . In particular Γ is a

regular graph with valency $k := b_0$. We define $k_i := |\Gamma_i(x)|$ for any vertex x and $i = 0, 1, \dots, D$. Then we have $k_0 = 1$, $k_1 = k$, $c_{i+1}k_{i+1} = b_i k_i$ ($0 \leq i \leq D-1$) and thus

$$k_i = \frac{b_1 \cdots b_{i-1}}{c_2 \cdots c_i} k \quad (1 \leq i \leq D). \quad (1)$$

A regular graph Γ on n vertices with valency k is called a strongly regular graph with parameters (n, k, λ, μ) if there are two non-negative integers λ and μ such that for any two distinct vertices x and y , $|\Gamma_1(x, y)| = \lambda$ if $d(x, y) = 1$ and μ otherwise. A connected non-complete strongly regular graph is just a distance-regular graph with diameter 2.

The adjacency matrix $A = A(\Gamma)$ is the $(|V(\Gamma)| \times |V(\Gamma)|)$ -matrix with rows and columns indexed by $V(\Gamma)$, where the (x, y) -entry of A is 1 if $d(x, y) = 1$ and 0 otherwise. The eigenvalues of Γ are the eigenvalues of $A = A(\Gamma)$. It is well-known that a distance-regular graph Γ with diameter D has exactly $D + 1$ distinct eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$ which are the eigenvalues of the following tridiagonal matrix

$$L_1 := \begin{pmatrix} 0 & k & & & & & \\ c_1 & a_1 & b_1 & & & & \\ & c_2 & a_2 & b_2 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & c_i & a_i & b_i & \\ & & & & \cdot & \cdot & \cdot \\ & & & & & c_{D-1} & a_{D-1} & b_{D-1} \\ & & & & & & c_D & a_D \end{pmatrix}$$

(cf. [2, p.128]). The standard sequence $\{u_i(\theta) \mid 0 \leq i \leq D\}$ corresponding to an eigenvalue θ is a sequence satisfying the following recurrence relation

$$c_i u_{i-1}(\theta) + a_i u_i(\theta) + b_i u_{i+1}(\theta) = \theta u_i(\theta) \quad (1 \leq i \leq D)$$

where $u_0(\theta) = 1$ and $u_1(\theta) = \frac{\theta}{k}$. Then the multiplicity of eigenvalue θ is given by

$$m(\theta) = \frac{|V(\Gamma)|}{\sum_{i=0}^D k_i u_i^2(\theta)} \quad (2)$$

which is known as *Biggs' formula* (cf. [2, Theorem 4.1.4]).

Let Γ be a distance-regular graph with valency k , n vertices and diameter D . For $i = 0, 1, \dots, D$, let A_i be the $\{0, 1\}$ -matrix with where $(A_i)_{xy} = 1$ if and only $d(x, y) = i$ for vertices x, y of Γ . Let \mathcal{A} be the Bose-Mesner algebra of Γ , i.e. the matrix algebra over the complex numbers generated by $A = A_1$. Then \mathcal{A} has as basis $\{A_0 = I, A_1 = A, A_2, \dots, A_D\}$. The algebra \mathcal{A} also has a basis of idempotents $\{E_0 = \frac{1}{n}J, E_1, \dots, E_D\}$. Define the Krein numbers q_{ij}^ℓ where $0 \leq i, j, \ell \leq D$ by $E_i \circ E_j = \frac{1}{n} \sum_{\ell=0}^D q_{ij}^\ell E_\ell$. It is known the Krein numbers are non-negative real numbers, see [2, Prop. 4.1.5]. We will also need the absolute bound. Let $\theta_0 = k > \theta_1 > \cdots > \theta_D$ be the distinct eigenvalues of Γ with respective multiplicities $m_0 = 1, m_1, \dots, m_D$. Then for $0 \leq i, j, \leq D$ we have

$$\sum_{\ell \in \{0, \dots, D\}, \text{ such that } q_{ij}^\ell \neq 0} m_\ell \leq \begin{cases} m_i m_j & \text{if } i \neq j \\ m_i(m_i + 1)/2 & \text{if } i = j. \end{cases}$$

This is called the absolute bound.

For a graph Γ , a partition $\Pi = \{P_1, P_2, \dots, P_t\}$ of $V(\Gamma)$ is called equitable if there are constants α_{ij} ($1 \leq i, j \leq t$) such that all vertices $x \in P_i$ have exactly α_{ij} neighbours in P_j . The α_{ij} 's ($1 \leq i, j \leq t$) are called the parameters of the equitable partition.

Let Γ be a distance-regular graph. For a set S of vertices of Γ , define $S_i := \{x \in V(\Gamma) \mid d(x, S) := \min\{d(x, y) \mid y \in S\} = i\}$. The number $\rho = \rho(S) := \max\{i \mid S_i \neq \emptyset\}$ is called the covering radius of S . The set S is called a completely regular code of Γ if the distance-partition $\{S = S_0, S_1, \dots, S_{\rho(S)}\}$ is equitable. The following result was first shown by Delsarte [6] for strongly regular graphs and extended by Godsil to the class of distance-regular graphs.

Lemma 2.1 (Delsarte-Godsil bound) *Let Γ be a distance-regular graph with valency $k \geq 2$, diameter $D \geq 2$ and smallest eigenvalue θ_{\min} . Let $C \subseteq V(\Gamma)$ be a clique with c vertices. Then*

$$c \leq 1 + \frac{k}{-\theta_{\min}},$$

with equality if and only if C is a completely regular code with covering radius $D - 1$.

A clique C with $\#C = 1 + \frac{k}{-\theta_{\min}}$, is called a Delsarte clique of Γ . It is known that parameters of a Delsarte clique as a completely regular code only depend on the parameters of Γ .

A distance-regular graph Γ is called a geometric distance-regular graph if Γ contains a set of Delsarte cliques \mathcal{C} , such that every edge of Γ lies in exactly one member C of \mathcal{C} .

Examples of geometric distance-regular graphs are for example the bipartite distance-regular graphs, the Johnson graphs, the Grassmann graphs, the Hamming graphs and the bilinear forms graphs. See [10], for more information on geometric distance-regular graphs.

A geometric distance-regular graph with valency k and diameter D is called a regular near $2D$ -gon if $c_i a_1 = a_i$ for $i = 1, 2, \dots, D$. A generalized $2D$ -gon of order (s, t) , where $s, t \geq 1$ are integers, is a regular near $(2D)$ -gon with valency $k = s(t + 1)$ and intersection number $c_{D-1} = 1$. A generalized 4-gon of order (s, t) is called a generalized quadrangle of order (s, t) and is denoted by $\text{GQ}(s, t)$. In similar fashion, a generalized 6-gon (respectively 8-gon) of order (s, t) is called a generalized hexagon (octagon) of order (s, t) and is denoted by $\text{GH}(s, t)$ ($\text{GO}(s, t)$).

3 Proof of Theorem 1.1

In this section, we give a proof of the valency bound Theorem 1.1.

Proof of Theorem 1.1:

As Γ is coconnected, Γ is not complete multipartite. Let m be the multiplicity of θ_{\min} . As Γ is not complete multipartite, we see that $k \leq \frac{(m-1)(m+2)}{2}$ holds, by [2, Thm 5.3.2].

We consider the standard sequence $u_0 = 1, u_1, \dots, u_D$ of $\theta = \theta_{\min}$. Then $u_1 = \theta/k$ and

$$u_{i+1} = \frac{(\theta - a_i)u_i - c_i u_{i-1}}{b_i} \quad (i = 1, 2, \dots, D-1),$$

and as θ is the smallest eigenvalue $(-1)^i u_i > 0$ for $i = 0, 1, \dots, D$, see [2, Cor. 4.1.2] We may assume that $k \geq 4\alpha^{-2}$, and hence $c_1 = 1 \leq (\alpha^2/4)k$. Let $\ell := \max\{i \mid 1 \leq i \leq D \text{ such that } c_i \leq \alpha^{i+1}2^{-i-1}k\}$. and let $p := \min\{\ell + 1, D\}$.

Claim 1

The number u_i satisfies $|u_i|\alpha^{-i} \geq 2^{-i}$ for $i = 0, 1, 2, \dots, p$.

Proof of Claim 1.

We will show it by induction on i . For $i = 0$, it is obvious, as $u_0 = 1$. For $i = 1$ we have $|u_1| = |\theta/k| \geq \alpha$, so the claim holds for $i = 1$.

Let $i \geq 2$. Then $u_i = \frac{(\theta - a_{i-1})u_{i-1} - c_{i-1}u_{i-2}}{b_{i-1}}$ holds. As $(-1)^i u_i > 0, c_{i-1} \leq \alpha^i 2^{-i}k, b_{i-1} \leq k, a_{i-1} \geq 0, |u_{i-2}| \leq 1, \theta \leq -\alpha k$ we see that

$$|u_i| \geq \frac{k\alpha|u_{i-1}| - \alpha^i 2^{-i}k}{k}.$$

By the induction hypothesis we obtain $|u_i|\alpha^{-i} \geq 2^{1-i} - 2^{-i} = 2^{-i}$. This shows the claim by induction. ■

Let $1 \leq q \leq p$ be such that k_q is maximal among k_0, k_1, \dots, k_p .

Claim 2

The number of vertices n of Γ satisfies $n \leq (D+1)2^{(q+1)(D-q)}\alpha^{(q+1)(q-D)}k_q$.

Proof of Claim 2.

If $q < p$, or if $q = D$, then $n = \sum_{i=0}^D k_i \leq (D+1)k_q$, as, then $k_q = \max\{k_i \mid 0 \leq i \leq D\}$ and hence the claim follows in this case.

So we may assume $p = q < D$. As $c_q > \alpha^{q+1}2^{-(q+1)}k$ and

$$k_{q+j} = k_q \frac{b_q b_{q+1} \dots b_{q+j-1}}{c_{q+1} c_{q+2} \dots c_{q+j}} < k_q k^j c_q^{-j} < k_q 2^{(q+1)j} \alpha^{-(q+1)j}$$

for $j = 0, 1, \dots, D - q$. As $n = \sum_{i=0}^D k_i$, Claim 2 follows. ■

Let $f(D, \alpha) := \max\{(D+1)2^{(q+1)(D-q)+2q}\alpha^{(q+1)(q-D)-2q} \mid q = 1, 2, \dots, D\}$.

By Biggs' formula, see [2, Thm 4.1.4], Claim 1 and Claim 2, we have

$$m = \frac{n}{\sum_{i=0}^D u_i^2 k_i} < \frac{n}{u_q^2 k_q} \leq (D+1)2^{(q+1)(D-q)}\alpha^{(q+1)(q-D)}\alpha^{-2q}2^{2q} \leq f(D, \alpha).$$

Hence $k \leq (f(D, \alpha) - 1)(f(D, \alpha) + 2)/2$. This shows the theorem with $\kappa(D, \alpha) = (f(D, \alpha) - 1)(f(D, \alpha) + 2)/2$. ■

4 Diameter 2

In this section we will determine the connected strongly regular graphs with valency $k \geq 2$ and smallest eigenvalue at most $-k/2$.

Proposition 4.1 *Let Γ be a non-complete non-bipartite connected strongly regular graph, valency $k \geq 2$ and smallest eigenvalue θ_{\min} satisfying $\theta_{\min} \leq -k/2$, then Γ is one of the following:*

1. *The pentagon with intersection array $\{2, 1; 1, 1\}$;*
2. *The Petersen graph with intersection array $\{3, 2; 1, 1\}$;*
3. *The folded 5-cube with intersection array $\{5, 4; 1, 2\}$;*
4. *The 3×3 -grid with intersection array $\{4, 2; 1, 2\}$;*
5. *The generalized quadrangle $GQ(2, 2)$ with intersection array $\{6, 4; 1, 3\}$;*
6. *The generalized quadrangle $GQ(2, 4)$ with intersection array $\{10, 8; 1, 5\}$;*
7. *A complete tripartite graph $K_{t,t,t}$ with $t \geq 2$, with intersection array $\{2t, t-1; 1, 2t\}$*

Before we show this proposition we recall the following classification of Seidel.

Theorem 4.2 (Seidel [11], see also [2, Thm 3.12.4(i)]) *Let Γ be a strongly regular graph and with second smallest eigenvalue -2 . Then Γ is one of the following graphs:*

1. *A Cocktail Party graph $K_{n \times 2}$, with $n \geq 2$;*
2. *A $t \times t$ -grid with $t \geq 2$;*
3. *A triangular graph $T(n)$ with $n \geq 4$;*
4. *The Petersen graph;*
5. *The Schläfli graph;*
6. *The Shrikhande graph;*
7. *One of the three Chang graphs;*
8. *The halved 5-cube*

From now on let Γ be a non-bipartite distance-regular graph with valency $k \geq 2$, diameter $D \geq 2$ and smallest eigenvalue $\theta_{\min} \leq -k/2$. If $a_1 \geq 1$, then, by Lemma 2.1, Γ has no 4-cliques and any triangle is a completely regular code.

Let us first consider the case $a_1 \geq 2$. Then any triangle $T = \{x, y, z\}$ is a completely regular code and any vertex u at distance 1 from T has exactly two neighbours in T . Let $A_{ab} := \{u \in V(\Gamma) \mid u \sim a, u \sim b\}$ where $a \neq b$ and $a, b \in \{x, y, z\}$. Then A_{ab} forms a coclique, as there are no 4-cliques by the Delsarte-Godsil bound, and if $\{a, b, c\} = \{x, y, z\}$, then each vertex of A_{ab} is adjacent to each vertex of A_{ac} . As the valency of x, y and z equals $\#A_{xy} + \#A_{xz}$, $\#A_{xy} + \#A_{yz}$, and $\#A_{xz} + \#A_{yz}$, respectively, it follows that Γ is the complete tripartite graph $K_{t,t,t}$ where $t = \#A_{xy} = \#A_{xz} = \#A_{yz}$. This shows:

Lemma 4.3 *Let Γ be a distance-regular graph with valency $k \geq 2$, diameter $D \geq 2$ and smallest eigenvalue θ_{\min} . If $\theta_{\min} \leq -k/2$ and $a_1 \geq 2$, then Γ is a complete tripartite graph $K_{t,t,t}$ for some $t \geq 2$.*

This shows that, if the distance-regular graph is coconnected, then $a_1 \leq 1$.

Now we are ready to give the proof of Proposition 4.1

Proof: Assume the graph is not bipartite. First let us discuss the case when θ_{\min} is not an integer. Then Γ has intersection array $\{2t, t; 1, t\}$ and smallest eigenvalue $\frac{-1-\sqrt{4t+1}}{2}$. Hence $\theta_{\min} \leq -k/2 = -t$ implies that $t \leq 2$, and we have that Γ is the pentagon as for $t = 2$, θ_{\min} is an integer. So from now we may assume that θ_{\min} is an integer. Let θ_1 be the other non-trivial eigenvalue of Γ . Then θ_1 is a non-negative integer. It follows that $c_2 - k = \theta_1 \theta_{\min} \leq -k\theta_1/2$. This implies that $\theta_1 \leq 1$. For $\theta_1 = 0$, we obtain the complete tripartite graphs, and for $\theta_1 = 1$, the complement of Γ has smallest eigenvalue -2 . These have been classified in Theorem 4.2 and by checking them we obtain the proposition. ■

5 A bound on c_2

In this section we will give a bound on c_2 . We first give the following result. This is a slight generalisation of [2, Prop. 4.4.6 (ii)]. We give a proof for the convenience for the reader, following the proof of [2]. Before we do this we need to introduce the following. Let Γ be a distance-regular graph with valency k with an eigenvalue θ , say with multiplicity m . Let $1 = u_0, u_1, \dots, u_D$ be the standard sequence of θ . Then there exists a map $\phi : V(\Gamma) \rightarrow \mathbf{R}^m : x \mapsto \bar{x}$, such that the standard inner product between u and v satisfies $\langle \bar{u}, \bar{v} \rangle = u_{d_\Gamma(u,v)}$.

Proposition 5.1 (Cf. [2, Prop. 4.4.6(ii)]) *Let Γ be a distance-regular graph with diameter $D \geq 2$, and valency $k \geq 2$. Assume Γ contains an induced $K_{r,s}$ for some positive integers r and s . Let θ be an eigenvalue of Γ , distinct from $\pm k$, with standard sequence $1 = u_0, u_1, \dots, u_D$. Then*

$$(u_1 + u_2)((r + s)\frac{1 - u_2}{u_1 + u_2} + 2rs) \geq 0 \quad (3)$$

and

$$(u_1 - u_2)((r + s)\frac{1 - u_2}{u_1 - u_2} - 2rs) \geq 0 \quad (4)$$

hold. In particular, if θ is the second largest eigenvalue, then

$$1 + \frac{b_1}{\theta + 1} \geq \frac{2rs}{r + s}$$

holds.

Proof: Let $X = \{x_1, x_2, \dots, x_r\}$ and $Y = \{y_1, y_2, \dots, y_s\}$ be the two color classes of the induced $K_{r,s}$. Let G be the Gram matrix with respect to the set $\{\bar{u} \mid u \in X \cup Y\}$. Now $\{X, Y\}$ is an equitable partition of G with quotient matrix Q where

$$Q = \begin{pmatrix} 1 + (r-1)u_2 & su_1 \\ ru_1 & 1 + (s-1)u_2 \end{pmatrix}.$$

Multiplying the first column of Q by s and the second column by r we obtain the matrix

$$Q' := \begin{pmatrix} s + s(r-1)u_2 & rsu_1 \\ rsu_1 & r + r(s-1)u_2 \end{pmatrix}.$$

As G is positive semi-definite, it follows that Q and Q' are both positive semi-definite and hence $(1 \ 1)Q'(1 \ 1)^T \geq 0$ and $(1, -1)Q'(1, -1)^T \geq 0$. Hence we obtain Equations (3) and (4). If θ is the second largest eigenvalue of Γ , then $1 > u_1 > u_2 > \dots > u_D$ (using that the largest eigenvalue of the matrix T of [2, p. 130] equals θ_1), and $1 + \frac{b_1}{\theta_1} = \frac{1-u_2}{u_1-u_2}$ both hold. This implies the in particular statement. \blacksquare

This leads us to the following result.

Lemma 5.2 *Let Γ be a non-bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 2$. If the smallest eigenvalue of Γ , θ_{\min} , is at most $-k/2$, then $a_1 \leq 1$ and $c_2 \leq 5 + a_1$.*

Proof: Let $\theta := \theta_{\min} \leq -k/2$. We already have established that if $a_1 \geq 2$, then the graph is complete tripartite and hence diameter is equal to 2. So this implies $a_1 \leq 1$. Let $1 = u_0, u_1, \dots, u_D$ be the standard sequence of Γ with respect to θ . Then $u_1 + u_2 = \frac{1}{kb_1}(\theta + k)(\theta + 1) < 0$. The induced subgraph of Γ consisting of two vertices at distance 2 and their common neighbours is a K_{2,c_2} . By Equation (3), we obtain that if $a_1 = 0$, then $3 > 1 - \frac{k-1}{\theta_{\min}-1} \geq \frac{4c_2}{2+c_2}$ and hence $c_2 \leq 5$. If $a_1 = 1$ then $\theta_{\min} = -k/2$ and $\sigma_1 = -1/2$ and $\sigma_2 = 1/4$ and again using Equation (3), we obtain $c_2 \leq 6$. This shows the lemma. \blacksquare

6 The case $a_1 = 1$

In this section we will discuss the situation for $a_1 = 1$. We will start with the following easy observation.

Proposition 6.1 *Let Γ be a distance-regular graph with valency $k \geq 3$, diameter $D \geq 2$, intersection number $a_1 = 1$, and smallest eigenvalue $\theta_{\min} \leq -k/2$. Then $\theta_{\min} = -k/2$, Γ is geometric, and there exists an integer i , $2 \leq i \leq D$, such that $a_j = c_j$ for $1 \leq j < i$, $a_i = k/2$ and $a_j = b_j$ for $i+1 \leq j \leq D$, with the understanding that $b_D = 0$. Moreover, if $a_D = k/2$, then Γ is a regular near $2D$ -gon of order $(2, k/2 - 1)$.*

Proof: As each triangle is a Delsarte clique, it follows that Γ is a geometric distance-regular graph. This implies the proposition. (For details, we refer to Koolen and Bang [10].) ■

In the following result, we summarise the known existence results about regular near $2D$ -gons with $a_1 = 1$.

Theorem 6.2 *Let $D \geq 2$. Let Γ be a regular near $2D$ -gon of order $(2, t)$. Then $c_2 \in \{1, 2, 3, 5\}$. Moreover, the following holds:*

1. *If $c_2 = 5$, then Γ is the dual polar graph of type ${}^2A_{2D-1}(2)$.*
2. *If $c_2 = 3$, then Γ is the dual polar graph of type $B_D(2)$, or if $D = 3$, the Witt graph associated to M_{24} (see [2, Thm 11.4.1]) with intersection array $\{30, 28, 24; 1, 3, 15\}$.*
3. *If $c_2 = 2$, then Γ is the Hamming graph $H(D, 3)$, or if $D = 3$, the coset graph of the truncated binary Golay code with intersection array $\{21, 20, 16; 1, 2, 12\}$.*
4. *If $c_2 = 1$ and $D = 3$, then Γ is one of the following:*
 - (a) *The generalized hexagon $GH(2, 1)$ with intersection array $\{4, 2, 2; 1, 1, 2\}$;*
 - (b) *The two generalized hexagons $GH(2, 2)$ with intersection array $\{6, 4, 4; 1, 1, 3\}$;*
 - (c) *The generalized hexagon $GH(2, 8)$ with intersection array $\{18, 16, 16; 1, 1, 9\}$.*
5. *If $c_2 = 1$ and $D = 4$, then Γ is one of the following:*
 - (a) *The generalized octagon $GO(2, 1)$ with intersection array $\{4, 2, 2, 2; 1, 1, 1, 2\}$;*
 - (b) *A generalized octagon $GO(2, 4)$ with intersection array $\{10, 8, 8, 8; 1, 1, 1, 5\}$;*
 - (c) *The Cohen-Tits regular near octagon associated with the Hall-Janko group (see [2, Thm 13.6.1]) with intersection array $\{10, 8, 8, 2; 1, 1, 4, 5\}$.*

Proof: For $D = 2$, see for example [8, Cor. 10.9.5]. For $D = 3$, see [4, Sect. 3.5]. For $D = 4$, this follows from [4, Sect. 3.6] and [5]. This shows the theorem for diameter at most 4. The cases $c_2 = 5$ and $c_2 = 4$ follows from Brouwer and Wilbrink [3] who classified the regular near $2D$ -gons having $D \geq 4$, $c_2 \geq 3$ and $a_1 \geq 1$, see also [12, Thm 9.11]. For $D \geq 3$, $c_2 \geq 2$, $c_3 = 3$ and $a_1 \geq 1$, it was shown by Van Dam et al. [12, Thm 9.11] that Γ is the Hamming graph $H(D, 3)$. For $D \geq 4$ and $c_2 = 2$, it is shown by Brouwer and Wilbrink that then also the regular near octagon with intersection array $\{2c_4, 2c_4 - 2, 2c_4 - 2c_2, 2c_4 - 2c_3; 1, c_2, c_3, c_4\}$ must exist and as by the diameter 4 case, we find that it must be the Hamming graph $H(4, 3)$, and hence $c_3 = 3$ holds. This shows

that if $c_2 = 2$ and $D \geq 4$ we must have the Hamming graph $H(D, 3)$. This finishes the proof of the theorem. \blacksquare

For $a_1 = 1$ and $\theta_{\min} = -k/2$, we can improve the valency bound of Theorem 1.1. Note that in Hiraki and Koolen [9] a similar bound was obtained for regular near polygons.

Proposition 6.3 *Let Γ be a distance-regular graph with a_1 , valency $k \geq 4$ and diameter $D \geq 2$. Then $k \leq 2^{2D+1} - 2$. Moreover, if $c_D = k$, then $k \leq 2^{2D-2} - 2$.*

Proof: Let (u_0, u_1, \dots, u_D) be the standard sequence corresponding to $\theta_{\min} = -k/2$. Let m be the multiplicity of θ_{\min} . By Proposition 6.1 there exists $2 \leq i \leq D$ be such that $a_i = k/2$. It is easy to show by induction, again using Proposition 6.1, that $u_j = (-2)^{-j}$ if $j \leq i$ and $u_j = (-2)^{j-2i}$ if $j \geq i$.

So, by Biggs' formula, $m \leq \frac{1}{\min\{u_i^2 | i=1,2,\dots,D\}} \leq 2^{2D}$. As $2m \geq k+2$, we find $k \leq 2^{2D+1} - 2$.

If $i \leq D-1$, then

$$k_{i+1} + k_{i-1} \geq \frac{b_i + c_i}{\max\{c_{i+1}, b_{i-1}\}} k_i \geq \frac{k/2}{k} k_i = k_i/2.$$

For positive real numbers a, b, c, d , if $a/c \geq b/d$ holds, then

$$\frac{a}{c} \geq \frac{a+b}{c+d} \geq \frac{b}{d}$$

holds. Using this, we see that

$$m = \frac{\sum_{j=0}^D k_j}{\sum_{j=0}^D k_j u_j^2} \leq \frac{k_{i-1} + k_i + k_{i+1}}{2^{-2i+2} k_{i-1} + 2^{-2i} k_i + k_{i+1} 2^{-2i+2}},$$

as $|u_j| \geq 2^{2-i}$ for $j \notin \{i-1, i, i+1\}$. Hence $m \leq \frac{3/2}{3 \times 2^{-2i}} = 2^{2i-1} \leq 2^{2D-3}$. As $k \leq 2m - 2$, by [9, Prop. 3], we see that $k \leq 2^{2D-2} - 2$ if $i \leq D-1$. This shows the proposition as $c_D = k$ if and only if $i \leq D-1$. \blacksquare

In view of Proposition 6.1 and Theorem 6.2, we only need to classify the distance-regular graphs with diameter D equals to 3 or 4, $a_1 = 1$ and $c_D = k$.

Theorem 6.4 *Let Γ be a distance-regular graph with diameter D equals 3 or 4, $a_1 = 1$ and valency k , $c_D = k$ and smallest eigenvalue $-k/2$. Then one of the following hold:*

1. $D = 3$ and Γ is a distance-regular graph with intersection array $\{8, 6, 1; 1, 3, 8\}$ (see [2, p. 224]), or the line graph of the Petersen graph with intersection array $\{4, 2, 1; 1, 1, 4\}$;
2. $D = 4$ and Γ is the distance-regular graph with intersection array $\{6, 4, 2, 1; 1, 1, 4, 6\}$ (see [2, Thm 13.2.1]).

Proof: As $a_1 = 1$, the valency k is even. By Proposition 6.3 we have for diameter 3 that the valency k is bounded by $k \leq 14$ and for diameter 4 we obtain $k \leq 62$. We generated all the possible intersection arrays of diameter 3 and 4 with $k \leq 14$ for diameter 3 and $k \leq 62$ for diameter 4, such that $c_2 \leq 6$, the c_i 's are increasing, the b_i 's are decreasing, the valencies k_i are positive integers, satisfying the conditions of Proposition 6.1, $c_d = k$ and the multiplicities of the eigenvalues are positive integers. Besides the intersection arrays in the theorem we obtained only the following two intersection arrays $\{10, 8, 3; 1, 2, 10\}$ and $\{12, 10, 3; 1, 3, 12\}$. As both have eigenvalue $-k/2$ with multiplicity 7 and the number of vertices equals 63, we find by the absolute bound (see [2, Prop. 4.1.5]) that if the graph exists, it must have a vanishing Krein parameter, but that is not the case. So there is no distance-regular graph with either of these two intersection arrays. This shows the theorem. \blacksquare

7 Diameter 3 and $a_1 = 0$

Note that there are infinitely many bipartite distance-regular graphs with diameter 3.

In the following result, we show that a non-bipartite distance-regular graph with diameter 3, valency k and smallest eigenvalue at most $-k/2$ has k at most 64.

Proposition 7.1 *Let Γ be a non-bipartite triangle-free distance-regular graph with diameter 3, valency k and smallest eigenvalue θ_{\min} at most $-k/2$. Then $k \leq 64$ holds.*

Proof: Let $(u_0 = 1, u_1, u_2, u_3)$ be the standard sequence with respect to $\theta := \theta_{\min}$. Let Γ have distinct eigenvalues $\theta_0 = k > \theta_1 > \theta_2 > \theta_3 = \theta$. Let

$$L = \begin{pmatrix} 0 & k & 0 & 0 \\ 1 & 0 & k-1 & 0 \\ 0 & c_2 & a_2 & b_2 \\ 0 & 0 & c_3 & a_3 \end{pmatrix}.$$

The matrix L has as eigenvalues $\theta_0, \theta_1, \theta_2, \theta_3$, and hence $\text{tr}(L^2) = \theta_0^2 + \theta_1^2 + \theta_2^2 + \theta_3^2 \geq k^2 + k^2/4$. On the other hand we have $\text{tr}(L^2) = a_2^2 + a_3^2 + 2k + 2c_2(k-1) + 2c_3b_2$. Replacing b_2 by $k - a_2 - c_2$ and a_3 by $k - c_3$, we obtain $\text{tr}(L^2) = k^2 + (c_3 - a_2)^2 + k(2 + 2c_2) - 2c_2(1 + c_3) \leq k^2 + (a_2 - c_3)^2 + 12k$ as $c_2 \leq 5$ by Lemma 5.2. This means that $(a_2 - c_3)^2 + 12k \geq k^2/4$. Now assume $k \geq 65$ to obtain a contradiction. Then $12k \leq (12/65)k^2$. This implies that $|a_2 - c_3| \geq (0.255)k$. This means that at least one of c_3 and a_2 is at least $(0.255)k$. We are going to estimate the multiplicity m of θ . On the one hand, $m \geq k$, as $a_1 = 0$. On the other hand, by Biggs' formula, we have

$$m = \frac{n}{\sum_{i=0}^3 k_i u_i^2},$$

where n is the number of vertices of Γ . This means that

$$m \geq \frac{1}{\min\{u_i^2 \mid i = 0, 1, 2, 3\}}.$$

We have $u_0 = 1$, $u_1 = \theta/k \leq -1/2$,

$$u_2 = \frac{\theta u_1 - 1}{k - 1} \geq \frac{k - 4}{4k - 4} \geq \frac{61}{256}.$$

So in order that $m \geq k \geq 65$ holds, we must have $u_3^2 < 1/64$, or, in other words, $u_3 > -1/8$. We obtain

$$(k - a_2)/8 > b_2/8 > -b_2 u_3 = (-\theta + a_2)u_2 + c_2 u_1 \geq (k/2 + a_2)(61/256) - 5,$$

and this implies $a_2 < \frac{3k+2560}{186}$. As $k \geq 65$, it follows that $a_2 < k/4$. As we have already established that at least one of a_2 and c_3 is at least $(0.255)k$, we find $c_3 \geq (0.255)k$. We find $-b_2 u_3 = u_2(a_2 - \theta) + c_2 u_1 \geq \frac{61}{256}(-\theta) + 5\frac{\theta}{k} = -\theta(\frac{61}{256} - \frac{5}{k}) \geq \frac{k}{2}(\frac{61}{256} - \frac{5}{k})$. As $b_2 < k$ and $k \geq 65$, we obtain $-u_3 \geq \frac{1}{2}(\frac{61}{256} - \frac{5}{65}) > 41/512$.

For positive real numbers a, b, c, d , if $a/c \geq b/d$ holds, then

$$\frac{a}{c} \geq \frac{a+b}{c+d} \geq \frac{b}{d}$$

holds. Using this, we see that

$$m = \frac{\sum_{i=0}^3 k_i}{\sum_{i=0}^3 k_i u_i^2} \leq \frac{k_2 + k_3}{u_2^2 k_2 + u_3^2 k_3}$$

holds, as $\frac{1+k}{1+ku_1^2} \leq 4$. Now

$$\frac{k_2 + k_3}{k_2 u_2^2 + k_3 u_3^2} = \frac{c_3 + b_2}{c_3 u_2^2 + b_2 u_3^2} \leq \frac{c_3 + k}{c_3 u_2^2 + k u_3^2},$$

as $|u_3| < 1/8 < u_2$, $k_3 = \frac{b_2}{c_3} k_2$ and $b_2 < k$ all hold. Using $c_3 \geq (0.255)k$, $u_2 \geq 61/256$ and $|u_3| \geq 41/512$, we find that

$$m \leq \frac{k_2 + k_3}{k_2 u_2^2 + k_3 u_3^2} \leq 64,$$

a contradiction. This shows the proposition. ■

Now we come to the main result of this section.

Theorem 7.2 *Let Γ be a non-bipartite distance-regular graph with diameter 3, $a_1 = 0$, valency $k \geq 2$ and smallest eigenvalue at most $-k/2$. Then Γ is one of the following:*

1. *The 7-gon, with intersection array $\{2, 1, 1; 1, 1, 1\}$;*
2. *The Odd graph with valency 4, O_4 , with intersection array $\{4, 3, 3; 1, 1, 2\}$;*
3. *The Sylvester graph with intersection array $\{5, 4, 2; 1, 1, 4\}$;*
4. *The second subconstituent of the Hoffman-Singleton graph with intersection array $\{6, 5, 1; 1, 1, 6\}$;*
5. *The Perkel graph with intersection array $\{6, 5, 2; 1, 1, 3\}$;*

6. *The folded 7-cube with intersection array $\{7, 6, 5; 1, 2, 3\}$;*
7. *A possible distance-regular graph with intersection array $\{7, 6, 6; 1, 1, 2\}$;*
8. *A possible distance-regular graph with intersection array $\{8, 7, 5; 1, 1, 4\}$;*
9. *The truncated Witt graph associated with M_{23} (see [2, Thm 11.4.2]) with intersection array $\{15, 14, 12; 1, 1, 9\}$;*
10. *The coset graph of the truncated binary Golay code with intersection array $\{21, 20, 16; 1, 2, 12\}$;*

Proof: By Proposition 7.1 we have that the valency k is bounded by $k \leq 64$. We generated all the possible intersection arrays of diameter 3 with $k \leq 64$, such that $a_1 = 0$, $c_2 \leq 5$, the c_i 's are increasing, the b_i 's are decreasing, the valencies k_i are positive integers, and the multiplicities of the eigenvalues are positive integers. Besides the intersection arrays listed in the theorem, we only found the intersection arrays $\{5, 4, 2; 1, 1, 2\}$ and $\{13, 12, 10; 1, 3, 4\}$. It was by Fon-der-Flaass [7] that there are no distance-regular graphs with intersection array $\{5, 4, 2; 1, 1, 2\}$. The intersection array $\{13, 12, 10; 1, 3, 4\}$ is ruled out by [2, Thm. 5.4.1]. This shows the theorem. ■

8 Proofs of Theorems 1.2 and 1.3

In this section we give the proof of Theorems 1.2 and 1.3.

Proof of Theorem 1.2: For diameter 2, it follows from Proposition 4.1. For $a_1 \neq 0$ and diameter 3 and 4, it follows from Proposition 6.1, and Theorems 6.2 and 6.4. For diameter 3 and $a_1 = 0$, it follows from Theorem 7.2. ■

Before we give the proof of Theorem 1.3, let us recall the chromatic number of a graph. A proper coloring with t colors of a graph Γ is a map $c : v(\Gamma) \rightarrow \{1, 2, \dots, t\}$ where t is a positive number such that $c(x) \neq c(y)$ for any edge xy . The chromatic number of Γ denoted by $\chi(\Gamma)$ is the minimal t such that there exists a proper coloring of Γ with t colors. We also say that such a graph is $\chi(\Gamma)$ -chromatic. An independent set of Γ is a set S of vertices such that there are no edges between them.

Hoffman showed the following result for regular graphs.

Lemma 8.1 (Hoffman bound), cf. [2, Prop. 1.3.2]. *Let G be a k -regular graph with n vertices and with smallest eigenvalue θ_{\min} . Let S be an independent set of Γ with s vertices. Then*

$$s \leq \frac{n}{1 + \frac{k}{-\theta_{\min}}}.$$

This means that if a k -regular graph Γ on n vertices is 3-chromatic, then it must have an independent set of size $n/3$ and by the Hoffman bound we find that the smallest eigenvalue of Γ is at most

$-k/2$. Now we are ready to give the proof for Theorem 1.3.

Proof of Theorem 1.3: (i): By above we only need to check the graphs of Theorem 1.2. The six graphs, we list, are shown to be 3-chromatic in [1, Section 3.4]. For the case $a_1 > 0$, it was shown that the last three graphs are the only 3-chromatic distance-regular graphs with diameter 3 in [1, Thm. 3.6]. So we only need to check the graphs with $a_1 = 0$. That the distance-regular graphs with intersection arrays $\{21, 20, 16; 1, 2, 12\}$ and $\{7, 6, 5; 1, 2, 3\}$ are not 3-chromatic follows from [1, Sect. 3.6] That the distance-regular graph with intersection array $\{15, 14, 12; 1, 1, 9\}$ is not 3-chromatic follows from [1, Sect. 3.7].

That the distance-regular graphs with intersection arrays $\{5, 4, 2; 1, 1, 4\}$, $\{6, 5, 1; 1, 1, 6\}$, $\{7, 6, 6; 1, 1, 2\}$, $\{8, 7, 5; 1, 1, 4\}$ are not 3-chromatic follows from [1, Sect. 3.9].

(ii): The two graphs we list are shown be 3-chromatic in [1, Sect. 3.4]. In [1, Thm. 3.3 & Prop. 3.8], it is shown that the Hamming graph $H(D, 3)$ is the only 3-chromatic distance-regular graph with $c_2 \geq 2$, $a_D > 0$ and $D \geq 4$. [1, Thm. 3.3] also shows that the distance-regular graph with intersection array $\{6, 4, 2, 1; 1, 1, 4, 6\}$ is not 3-chromatic, as it has induced pentagons. That the regular near octagon associated with the Hall-Janko group (see [2, Thm 13.6.1]) with intersection array $\{10, 8, 8, 2; 1, 1, 4, 5\}$, is not 3-chromatic is shown on [1, p. 299]. That a generalized octagon $GO(2, 4)$ with intersection array $\{10, 8, 8, 8; 1, 1, 1, 5\}$ is not 3-chromatic, follows from [1, Thm. 3.2]. This shows Theorem 1.3. ■

9 Open problems

Now we give some open problems.

1. Classify the geometric distance-regular graphs with intersection number $a_1 = 1$.
2. Finish the classification of the non-bipartite distance-regular graphs with diameter 4, valency k and smallest eigenvalue at most $-k/2$.
3. Classify the non-bipartite distance-regular graphs with diameter 3, valency k and smallest eigenvalue $-k/3$.

References

- [1] A. Blokhuis, A.E. Brouwer, and W.H. Haemers. On 3-chromatic distance-regular graphs. *Des. Codes Cryptogr.*, 44:293–305, 2007.
- [2] A.E. Brouwer, A.M. Cohen, and A. Neumaier. *Distance-Regular Graphs*. Springer-Verlag, Berlin, 1989.

- [3] A.E. Brouwer and H.A. Wilbrink. The structure of near polygons with quads. *Geom. Dedicata*, 14:145–176, 1983.
- [4] B. De Bruyn. *Near Polygons*. Birkhäuser, Basel, 2006.
- [5] B. De Bruyn. The nonexistence of regular near octagons with parameters $(s, t, t_2, t_3) = (2, 24, 0, 8)$. *Electron. J. Combin.*, 17:R149, 2010.
- [6] P. Delsarte. *An algebraic approach to the association schemes of coding theory*, volume 10 of *Philips Res. Reports Suppl.* 1973.
- [7] D.G. Fon-Der-Flaass. There exists no distance-regular graph with intersection array $(5, 4, 3; 1, 1, 2)$. *European J. Combin.*, 14:409–412, 1993.
- [8] C.D. Godsil and G.F. Royle. *Algebraic Graph Theory*. Springer, New York, 2001.
- [9] A. Hiraki and J.H. Koolen. A Higman-Haemers inequality for thick regular near polygons. *J. Algebraic Combin.*, 20:213–218, 2004.
- [10] J.H. Koolen and S. Bang. On distance-regular graphs with smallest eigenvalue at least $-m$. *J. Combin. Theory Ser. B*, 100:573–584, 2010.
- [11] J.J. Seidel. Strongly regular graphs with $(-1, 1, 0)$ adjacency matrix having eigenvalue 3. *Lin. Alg. Appl.*, 1:281–298, 1968.
- [12] E.R. van Dam, J.H. Koolen, and H. Tanaka. Distance-regular graphs. [arXiv:1410.6294v1].